JOURNAL OF AIRCRAFT
Vol. 32, No. 2, March–April 1995

Multiple Order Pole Pure Lag Rational Function Approximations for Unsteady Aerodynamics

P. M. Mujumdar* and N. Balan†
Indian Institute of Technology, Bombay 400076, India

Most existing rational function approximations for the time domain representation of unsteady generalized airloads lead to an ill-conditioned system dynamic matrix in the presence of closely spaced poles. A new class of multiple order pole pure lag rational function approximations (RFA) is presented in this article to overcome this problem. The present class of approximations is developed as a consistent generalization of an existing simple pole pure lag RFA while preserving the resulting state vector dimension. A nonlinear nongradient optimization technique is used for the computation of the lag poles in the approximation. The structure of the proposed class of approximations preserves the pure lag form, thus allowing for specific physical interpretations of the individual terms in the approximation and renders the subsequent optimization problem simpler, in that fewer constraints need to be imposed during the optimization process. Furthermore, the new class of approximations leads to substantial reduction in computational costs for optimization for a given fit accuracy, in comparison to existing simple pole approximations. Results are presented for the case of a typical delta wing fighter aircraft configuration.

Introduction

MODERN flexible fighter aircraft may often possess one or more natural frequencies of structural vibration modes that fall within the control bandwidth of the aircraft control system. This is especially true of unstable fighter aircraft configurations with stability augmentation systems. As a result, in order to carry out an aeroservoelastic analysis, it becomes necessary to model the aircraft as a flexible structure, and include the effects of unsteady aerodynamics in the flexible aircraft model. The equations of motion for a flexible aircraft model can be written as

$$[M]\ddot{\xi} + [C]\dot{\xi} + [K]\xi = [F(t)] \tag{1}$$

where $[M]$, $[C]$, and $[K]$ are the modal generalized mass, damping, and stiffness matrices, respectively, ξ is the vector of modal generalized coordinates, and $[F(t)]$ is the vector of modal generalized unsteady airloads consisting of a combination of external aerodynamic forces such as gust loads and aerodynamic forces developed due to motion along the generalized coordinates. This article deals exclusively with forces of the latter type. Hence, the vector $[F(t)]$ can be expressed as a function of the generalized coordinates in the form $[F(t)] = F(\xi, \dot{\xi}, \ddot{\xi})$.

Computation of the unsteady airloads is typically carried out using techniques like the doublet lattice method, the Mach box method, the kernel function method, etc. A major limitation of these techniques is that the unsteady airloads can be calculated only for the case of harmonic motion, i.e., along the imaginary axis of the complex Laplace plane. Further, computation of these airloads is carried out only in the frequency domain, for discrete values of the reduced frequency. For this purpose, the unsteady airloads are written in the frequency domain as $[F(\iota\bar{k})] = q_d[Q(\iota\bar{k})]\xi$, where q_d is the dynamic pressure, $\bar{k}$ is the reduced frequency, and $[Q(\iota\bar{k})]$ is the (complex valued) matrix of unsteady aerodynamic influence coefficients, henceforth, termed the unsteady aerodynamic matrix. In the absence of a viable technique for the direct computation of unsteady airloads in the time domain, as required for aircraft response calculations, it becomes necessary to convert the unsteady airloads from the frequency domain to the time domain.

RFAs provide a convenient means of carrying out this conversion, via an intermediate step wherein the frequency domain airloads are converted to the Laplace domain. The use of RFAs permits each element of the unsteady aerodynamic matrix to be written as a rational function, i.e., as a ratio of polynomials, in the Laplace variable s. The structure of the RFA then allows for easy conversion from the Laplace domain to the time domain. The primary advantage of RFAs stems from the linear time invariant (LTI) form of the resulting equations of motion, thereby allowing the use of existing and efficient solution procedures for linear systems. The presence of the unsteady airloads is reflected in the time domain in terms of an augmented state vector, which, in addition to the generalized displacement and velocity, also contains additional aerodynamic states, termed aerodynamic lag states.

Early studies on the conversion of unsteady aerodynamic loads from the frequency domain to the time domain were based on the RFA described by the equation

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L} A_{(l+2)_{jk}} \frac{\bar{s}}{(\bar{s} + \beta_l)} \tag{2}$$

where $\hat{Q}_{jk}$ is the approximate value of the aerodynamic influence coefficient from the jth row and kth column of the unsteady aerodynamic matrix, $\bar{s} = sb/U_\infty$ is the nondimensionalized Laplace variable, $A_{p_{jk}}(p = 0, 1, 2, \ldots, 2 + N_L)$ are matrix coefficients in the RFA, and β_l are the lag poles in the RFA. It can be seen that this RFA uses the same denominator coefficients, termed lag poles, for all elements of the unsteady aerodynamic matrix. RFAs of this form will henceforth be termed as column-independent RFAs in this article. Roger[1] and Abel[2] were among the first to employ such an RFA for the conversion of unsteady airloads to the time domain. They used a least-squares technique for the computation of the linear parameters $A_{p_{jk}}(p = 0, 1, 2, \ldots, 2 + N_L)$ in the RFA, and this formulation is termed the conventional least-squares (CLS) RFA. However, the accuracy of the resulting fit was seen to be dependent on the choice of lag poles, for which no systematic procedure was outlined.

Received Nov. 29, 1993; revision received July 5, 1994; accepted for publication July 6, 1994.

*Associate Professor, Department of Aerospace Engineering.
†Doctoral Student, Department of Aerospace Engineering.

A variation of the CLS RFA is the modified matrix Padé (MMP) RFA,[3] which differs from the former in that it uses the same lag poles for each element in a given column of the unsteady aerodynamic matrix, corresponding to a given degree of freedom (DOF) for the aircraft model under consideration, but the number of lag poles and their numerical values are allowed to vary between columns. This type of RFA will henceforth be termed as the column-dependent RFA. For the same number of lag poles, the MMP RFA yields a state vector of the same dimension as the CLS RFA. However, as in the case of the CLS RFA, there was no systematic procedure for the numerical computation of the lag poles in the MMP RFA.

The minimum state (MS) method of Karpel,[4] as opposed to the CLS and MMP RFAs, starts with a state-space representation of the aircraft equations of motion, and yields the minimum number of aerodynamic lag states for a given number of lag poles, among the above three RFAs. Tiffany and Adams[5] introduced a constrained optimization technique for the computation of the lag poles in the CLS, MMP, and MS RFAs, such that the resulting least-squares fit error was minimized. The corresponding methods were termed the extended least-squares (ELS) RFA, the extended modified matrix Padé (EMMP) RFA, and the extended minimum state (EMS) RFA, respectively.

Hoadley and Karpel[6] demonstrated the application of minimum state unsteady aerodynamic approximations to an aeroservoelastic model in order to develop low-order state-space equations for control system analysis and design. Karpel[7] introduced the concept of physical weighting of the unsteady aerodynamic influence coefficients according to the incremental error of each coefficient on the system aeroelastic characteristics. This weighting was seen to yield a better fit of the more important terms at the expense of less important ones. Nissim[8] dealt with the problem of reduction of aerodynamic augmented states in active flutter suppression systems by identifying and separating the vibration modes into two categories: 1) those that have a significant influence on the flutter characteristics, and 2) the remaining modes. The aerodynamic influence coefficients corresponding to the former set of modes were approximated using the CLS RFA, while the latter were approximated using the quasisteady approximation. Karpel[9] provides a survey of various size reduction techniques for aeroservoelastic models.

In the process of examining an RFA similar to the ELS RFA, Eversman and Tewari[10] encountered the phenomenon of repeated poles, wherein two or more lag poles for a given column have values very close to each other, resulting in an ill-conditioned system dynamic matrix in the state-space form of the aircraft equations of motion. Consequently, they proposed a multiple order pole RFA, which yields the same order of fit error for the same state vector dimension as their original RFA, but results in a well-conditioned system dynamic matrix.

In the above RFAs, no direct physical interpretation can be attributed, in general, to any of the coefficient matrices $[A_0]$, $[A_1]$, . . . , when the lag terms are included. Suryanarayan et al.[11] have dealt with a restructured form of the CLS RFA, termed the pure lag RFA, which allows for the interpretation of $[A_1]$ as the quasisteady approximation to the aerodynamic damping matrix, and of $[A_2]$ as the aerodynamic inertia matrix. The successive matrices from $[A_3]$ onwards can then be interpreted as those representative purely of the unsteady aerodynamic effects arising from the shed vortices in the wake. The aerodynamic lag terms in the restructured form of the CLS RFA proposed in Ref. 11 are referred to as pure lag terms. However, Ref. 11 dealt with the selection of the lag poles in an ad-hoc manner.

The present study examines the implication of using an optimization technique with the pure lag RFA for the systematic computation of the pure lag poles. The phenomenon of repeated poles and consequent ill-conditioning of the system dynamic matrix is observed to be manifest in the process of optimized computation of the pure lag poles. A new class of multiple order pole pure lag (MPPL) RFAs is then proposed in this article, which overcome these deficiencies, while retaining the benefits of the pure lag representation as well as preserving the state vector dimension.

Simple Pole Pure Lag RFA

Formulation

The pure lag RFA was first proposed by Suryanarayan et al., and as given in Ref. 11, is described by the equation

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L} A_{(l+2)_{jk}} \frac{\bar{s}^3}{\beta_l^2(\bar{s} + \beta_l)} \tag{3}$$

This RFA is termed the column-independent simple pole pure lag (SPPL) RFA hereafter. It may be noted that, in the CLS RFA, the numerical values of the matrices $[A_1]$ and $[A_2]$ are dependent on the number of lag terms and lag coefficients and their values. In contrast, in the SPPL RFA, which is a restructured form of the CLS RFA, the matrices $[A_1]$ and $[A_2]$ are independent of the number of lag terms and their values. As a consequence, they can be written in terms of the exact values of the elements of the unsteady aerodynamic matrix and their derivatives at $\bar{s} = \imath\bar{k} = 0$ as

$$A_{0_{jk}} = \lim_{\bar{k}\to 0} \text{Re } \hat{Q}_{jk}(\imath\bar{k}) \tag{4}$$

$$A_{1_{jk}} = \lim_{\bar{k}\to 0} \frac{\partial}{\partial \bar{k}} \text{Im } \hat{Q}_{jk}(\imath\bar{k}) \tag{5}$$

$$A_{2_{jk}} = -\frac{1}{2} \lim_{\bar{k}\to 0} \frac{\partial^2}{\partial \bar{k}^2} \text{Re } \hat{Q}_{jk}(\imath\bar{k}) \tag{6}$$

The column-dependent form of the SPPL RFA can be described by the equation

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L(k)} A_{(l+2)_{jk}} \frac{\bar{s}^3}{\beta_{lk}^2(\bar{s} + \beta_{lk})} \tag{7}$$

In order to place the SPPL RFA on equal footing with the ELS and EMMP RFA, it becomes necessary to compute the numerical values of the pure lag coefficients by the minimization of a suitable error function. For this purpose, it is first observed that the matrices $[A_0]$, $[A_1]$, and $[A_2]$ in the SPPL RFA can be computed by the use of Eqs. (4), (5), and (6). Thus, these equations render superfluous the need to impose equality constraints on the approximation at $\bar{k} = 0$, as is done in earlier studies,[5] in order to improve the curve fit near $\bar{k} = 0$. This property is characteristic of all pure lag RFA representations. Furthermore, as explained in Ref. 11, Eq. (3) is meaningful even in a truncated form, for an appropriate limited range of reduced frequency. For example, in order to carry out a fit for the unsteady aerodynamic matrix at low reduced frequency values, it suffices to retain the first two terms in the RFA described by Eq. (3), after carrying out the computations for $N_L \geq 1$. In addition, the range of reduced frequency over which the fit has to be carried out can be progressively increased by the addition of successive terms to the RFA.

In the absence of an analytical expression for $Q_{jk}(\imath\bar{k})$, it becomes necessary to evaluate the derivatives of the elements of the unsteady aerodynamic matrix near $\bar{k} = 0$ by some appropriate numerical technique. This, in turn, requires knowledge of the values of the elements of the unsteady aerodynamic matrix at many points in the vicinity of $\bar{k} = 0$ for accurate evaluation of the derivatives occurring in Eqs. (5) and (6).

In this article the computation of the remaining linear parameters $[A_{(p+2)}]$ and the nonlinear parameters in the RFA is carried out using an iterative procedure involving a least-squares solution for the linear parameters in conjunction with a nonlinear optimization procedure for the nonlinear parameters. For this purpose, a quadratic cost function is defined for each individual element of the unsteady aerodynamic matrix in terms of the error between the exact values of that element of the aerodynamic matrix and the corresponding approximate values, at each reduced frequency as

$$\varepsilon_{jk} = \sum_{\nu=1}^{m} \frac{|Q_{jk}(\iota\bar{k}_\nu) - \hat{Q}_{jk}(\iota\bar{k}_\nu)|^2}{\max\{|Q_{jk}(\iota\bar{k}_\nu)|^2, 1\}} \tag{8}$$

with the summation in the above equation being carried out over all discrete reduced frequency locations in the range of interest.

The linear parameters are now obtained as the solutions of the set of linear algebraic equations defined by

$$\frac{\partial\varepsilon_{jk}}{\partial A_{(l+2)jk}} = 0, \qquad l = 1, \ldots, N_L(k) \tag{9}$$

In the absence of sufficient data near $\bar{k} = 0$ for the computation of one or more of the parameters $[A_0]$, $[A_1]$, and $[A_2]$, these parameters can also be included as variables to be determined using the least-squares technique, along with $[A_{(l+2)}]$ $[l = 1, \ldots, N_L(k)]$.

In order to compute the nonlinear parameters, an appropriate objective function is defined as a weighted sum of the errors ε_{jk} for each element of the unsteady aerodynamic matrix.

Accordingly, in the column-dependent case, the objective function is defined as

$$E_k = \sum_{j=1}^{N_\xi} W_{jk}\varepsilon_{jk} \tag{10}$$

while, in the column-independent case, the objective function is defined as

$$E = \sum_{j=1}^{N_\xi}\sum_{k=1}^{N_\xi} W_{jk}\varepsilon_{jk} \tag{11}$$

where N_ξ is the order of the unsteady aerodynamic matrix, and W_{jk} is an appropriate weighting factor for each element of the unsteady aerodynamic matrix, taken as unity in this study.

Finally, the optimization problem is posed mathematically in the column-dependent case as

$$\text{minimize } E_k(\text{w.r.t. } \beta_k) \quad \text{subject to } \beta_k > 0 \tag{12}$$

and in the column-independent case as

$$\text{minimize } E(\text{w.r.t. } \beta) \quad \text{subject to } \beta > 0 \tag{13}$$

Assuming equal number of lag terms N_L per column, the state-space form of the aircraft equations of motion in the column-dependent case are derived as follows. The state vector is defined as

$$X = \{\xi\ \dot{\xi}\ \xi_{L_1}\ \xi_{L_2} \cdots \xi_{L_{N_L}}\}^T \tag{14}$$

where the aerodynamic lag vector ξ_{L_l}, $l = 1, \ldots, N_L$ is defined in terms of the vector of generalized coordinates as

$$\xi^{(k)}_{L_l}(s) = \frac{\bar{s}^2}{\beta^2_{lk}(\bar{s} + \beta_{lk})}\,\xi^{(k)}(s) \qquad (l = 1, \ldots, N_L) \tag{15}$$

the superscript k standing for the kth component of the vector.

Upon conversion of the above equation to the time domain, the governing equation for the lag states can be written as

$$\beta^2_{lk}\lambda\dot{\xi}^{(k)}_{L_l} + \beta^3_{lk}\xi^{(k)}_{L_l} = \lambda^2\ddot{\xi}^{(k)} \qquad (l = 1, \ldots, N_L) \tag{16}$$

where $\lambda = b/U_\infty$.

The aircraft equations of motion can now be written in the time domain as

$$[M]\ddot{\xi} + [C]\dot{\xi} + [K]\xi = q_d\left\{[A_0]\xi + \lambda[A_1]\dot{\xi} + \lambda^2[A_2]\ddot{\xi} + \lambda\sum_{l=1}^{N_L}[A_{(l+2)}]\dot{\xi}_{L_l}\right\} \tag{17}$$

The state-space form of the aircraft equations of motion can be written in the form $\dot{X} = [A]X$, where the dimension of the system dynamic matrix $[A]$ is given as $(2 + N_L)N_\xi$. An explicit expression for the matrix $[A]$ is given in the Appendix.

A similar procedure can also be used to obtain the state-space form of the aircraft equations of motion for the column-dependent case having a variable number of lag terms per column, while the state space equations for the column-independent case can be obtained by dropping k in Eqs. (15) and (16).

Results

The modal unsteady airloads considered in this study are typical of the longitudinal motion of a delta wing fighter aircraft, over a frequency range that is dictated by the actuator bandwidth. The aircraft is modeled with 7 DOF, viz., two rigid modes, q_1 and q_2 (corresponding to heave and pitch); four symmetric elastic modes, q_3, q_4, q_5, and q_6 (normal modes corresponding primarily to first and second wing bending, wing torsion, and fuselage bending); and one symmetric control surface mode q_7. The modal generalized unsteady airloads were calculated for symmetric level flight at sea level, for a Mach number of 0.9, at 25 equally spaced reduced frequency locations in the range $\bar{k} = 0$ to $\bar{k} = 0.3$. These modal unsteady airloads were computed by the doublet lattice method, using the aeroelasticity module of the ELFINI© (AMD-BA, France), general-purpose finite element software package.

The quasisteady aerodynamic stiffness matrix $[A_0]$ is computed from Eq. (4), while the quasisteady aerodynamic damping matrix $[A_1]$ is computed from a finite difference version of Eq. (5), using the values of the unsteady aerodynamic matrix at $\bar{k} = 0$ and $\bar{k} = 0.0005$. The matrix of aerodynamic inertia coefficients $[A_2]$ is included as an additional variable to be computed through the least-squares technique, as opposed to its computation through Eq. (6).

It can be seen from Eqs. (3) or (7) that the square of the pure lag pole in the denominator of each term under the summation symbol is independent of $\bar{s}$, and can be absorbed into the accompanying matrix coefficient, resulting in a simpler form of the pure lag RFA. For example, in the column-dependent case, the resulting RFA can be described by the equation

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L(k)} A_{(l+2)_{jk}}\frac{\bar{s}^3}{\bar{s} + \beta_{lk}} \tag{18}$$

Table 1 Pure lag pole values obtained for the SPPL RFA

	Pure lag numerical values						
Column	One lag pole	Two lag poles		Three lag poles		Four lag poles	
1	0.3276	0.5958	0.5959	0.2156	0.2164	0.3351	0.3409
				0.2174		0.3479	0.3650
2	0.1216	0.2529	0.2530	0.3731	0.3751	0.2290	0.2340
				0.3760		0.2359	0.2510
3	0.1602	0.2977	0.2978	0.4930	0.4960	0.6800	0.7000
				0.4970		0.7200	0.7600
4	0.3664	0.1574	0.1575	0.2714	0.2724	0.4039	0.4110
				0.2735		0.4200	0.4600
5	0.5366	0.2412	0.2413	0.1711	0.1717	0.2230	0.2270
				0.1726		0.2330	0.2450
6	0.1678	0.3136	0.3137	0.1890	0.1895	0.2710	0.2790
				0.1902		0.2860	0.2949
7	0.1831	0.4028	0.4030	0.7641	0.7680	2.4591	2.7791
				0.7740		3.1989	3.8910

A nonlinear nongradient univariate optimization technique[12] was used for the computation of the lag poles. In keeping with Eqs. (12) and (13), the design variables were constrained to be positive. It was observed that the use of the RFAs described by Eqs. (7) and (18) led to the same numerical values of the resulting lag poles and the least-squared fit error defined by Eq. (10). A similar observation was also made for the column-independent case.

Numerical results were generated by varying the number of terms under the summation symbol in Eq. (18). These results showed that for a given number of lag terms per column, the numerical values of the lag poles so obtained were very close to each other. This is reflected in Table 1, which shows the numerical values of the lag poles computed for all columns using the column-dependent formulation, for varying $N_L(k)$. For example, it is seen from Table 1 that the use of four lag terms for column 1 led to four pure lag poles lying very close to each other, with numerical values 0.3351, 0.3409, 0.3479, and 0.3650. Similar results were observed for the other columns as well, with the only exception being column 7, for the case of four pure lag terms in the approximation. The column-independent case was also observed to lead to similar results.

In order to validate the accuracy of the column-dependent SPPL RFA, the approximate aerodynamic influence coefficients $\hat{Q}_{jk}(i\bar{k})$ given by Eq. (18) with two lag states per column, were used in the computation of the open loop frequency response in terms of the normal acceleration n_z and the pitch rate q at the sensor locations. Two different methods were used for the computation of the n_z response from the approximate aerodynamic influence coefficients. In the first method the computations were carried out by putting $s = i\bar{k}$ and $\xi = \xi_0(\omega)e^{i\omega t}$ in the modal equations [Eq. (17)]. In the second method, the n_z response was obtained from the state-space representation of the aircraft equations of motion. The n_z response computations carried out using the above two methods were compared with the exact n_z values obtained directly using the exact aerodynamic influence coefficients $Q_{jk}(i\bar{k})$. Figure 1 shows a comparison of the resulting errors in the magnitude and phase of the normal acceleration values with respect to the exact n_z values. It is clear from Fig. 1 that the approximate n_z response is accurate only when the frequency response computations are carried out from the modal equations, and that the use of the system dynamic matrix for the computation of n_z leads to highly inaccurate results. A similar trend was also observed for the pitch rate response obtained using the column-dependent SPPL RFA, as well as for the normal acceleration and pitch rate response obtained using the column-independent SPPL RFA.

The large error in the normal acceleration and pitch rate response obtained from the system dynamic matrix can be attributed to the existence of repeated poles as discussed above, resulting in an ill-conditioned system dynamic matrix. These inaccurate results are a consequence of the use of such an ill-conditioned matrix for the computation of n_z.

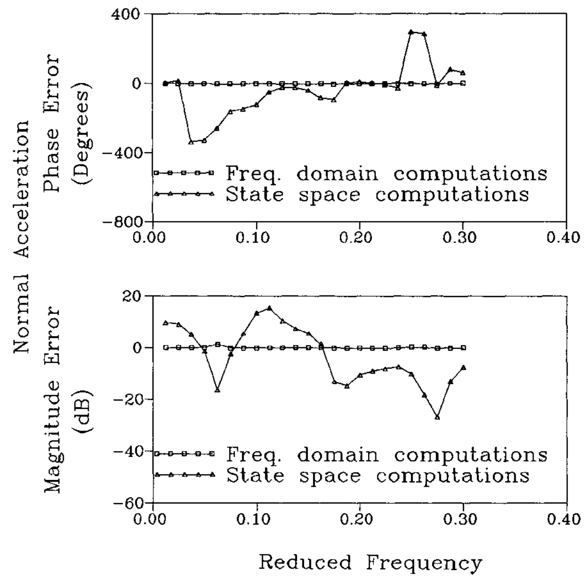

Fig. 1 Errors in normal acceleration response obtained from the column-dependent SPPL RFA using various techniques.

The results emphasize that it may not be sufficient to examine the errors in the response computed from the modal equations alone, to establish the suitability of the RFA for state-space modeling.

Multiple Pole Pure Lag

The existence of lag poles with numerical values very close to each other in a large number of cases in the SPPL RFA suggested the possibility of existence of multiple order lag poles in the RFA representation. Furthermore, in the event that two or more pure lag poles for the same column of the unsteady aerodynamic matrix have exactly the same numerical values, the RFA representation as given by Eq. (18) is no longer amenable to a unique least-squares solution. Hence, an alternate form of RFA representation, involving multiple order lag poles is called for, at least for those cases wherein the phenomena of repeated poles is observed. The objective of this study was to develop such a multiple order pole formulation, as a generalization of the SPPL RFA, which would preserve the pure lag form of the representation, and leave the state vector dimension unchanged. Towards this end, a new class of MPPL RFAs is proposed, the column-dependent form of which can be described by the generic equation

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L(k)} A_{(l+2)_{jk}} \frac{\bar{s}^{[3+p(l,k)]}}{(\bar{s}+\beta_k)^l} \tag{19}$$

where $p(l, k)$ is any integer valued function satisfying the conditions

$$p(1, k) = 0 \tag{20}$$

$$0 \le p(l, k) - p(l-1, k) \le 1, \qquad [2 \le l \le N_L(k)] \tag{21}$$

Equation (20) ensures that the numerator of the term multiplying the matrix $[A_3]$ is s^3, thereby retaining the pure lag form and preserving the state vector dimension up to $N_L(k) = 1$. Equation (21) ensures that each successive lag state is related to the previous one in the time domain by an ordinary differential equation of order of at the most 1, thereby preserving the state vector dimension for arbitrary $N_L(k) > 1$.

The column-independent form of the above class of RFAs can be described by the generic equation

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L} A_{(l+2)_{jk}} \frac{\bar{s}^{[3+p(l)]}}{(\bar{s}+\beta)^l} \tag{22}$$

where $p(l)$ satisfies the column-independent versions of Eqs. (20) and (21), and is independent of k.

It may be noted again that the term β_{lk}^2 or β_l^2 in the original form of the pure lag RFA as given by Eqs. (7) or (3), respectively, need not be retained, and has been absorbed into the coefficient $A_{(l+2)_{jk}}$ in the above MPPL term definitions as given by Eqs. (22) or (19), respectively.

It can be easily seen that the RFAs described by Eqs. (19) and (22) are pure lag RFA representations, since they satisfy Eqs. (4–6). Furthermore, for the case when the summation in Eq. (19) or (22) extends only over one term, i.e., $N_L(k)$ in Eq. (19) or N_L in Eq. (22) is equal to 1, then all the above MPPL RFA representations are equivalent to the SPPL RFA with one pure lag pole. This is a consequence of Eq. (20) or its column-independent version. For a fixed $N_L(k)$ or for a fixed N_L, the number of possible MPPL RFA representations is determined by the number of integer-valued functions $p(l, k)$ or $p(l)$ satisfying Eqs. (20) and (21), or their column-independent versions, respectively.

The procedure to obtain a state-space representation of the aircraft equations of motion that is consistent with the above new class of MPPL RFAs, is illustrated below for the column-dependent case, assuming equal pure lag pole multiplicity N_L per column. Thus, the state vector is defined as

$$X = \{\xi\ \dot{\xi}\ \xi_{L_1}\ \xi_{L_2} \cdots \xi_{L_{N_L}}\}^T \tag{23}$$

The first aerodynamic lag vector ξ_{L_1} is defined in terms of the vector of generalized coordinates in the nondimensionalized Laplace domain as

$$\xi_{L_1}^{(k)}(\bar{s}) = \frac{\bar{s}^2}{(\bar{s}+\beta_k)}\, \xi^{(k)}(\bar{s}) \tag{24}$$

where, as before, the superscript (k) refers to the kth coordinate of the vector. The second and successive aerodynamic lag states can be defined in the Laplace domain in terms of their immediate predecessors, and depends on the specific function $p(l, k)$ being used in the MPPL RFA representation. Thus

$$\begin{aligned} \xi_{L_l}^{(k)} &= \bar{s}\xi_{L_{(l-1)}}^{(k)}/(\bar{s}+\beta_k) \quad \text{if} \quad p(l, k) - p(l-1, k) = 1 \\ \xi_{L_l}^{(k)} &= \xi_{L_{(l-1)}}^{(k)}/(\bar{s}+\beta_k) \quad \text{if} \quad p(l, k) - p(l-1, k) = 0 \end{aligned} \tag{25}$$

where $2 \le l \le N_L$.

The time domain equations of motion for the aircraft modeled as a flexible structure are given by Eq. (17), where the lag states can be defined in the time domain in terms of the differential equations obtained by inversion of Eqs. (24) and (25) from the Laplace domain to the time domain. These definitions are given as

$$\lambda\dot{\xi}_{L_1}^{(k)} + \beta_k\xi_{L_1}^{(k)} = \lambda^2\ddot{\xi}^{(k)} \tag{26}$$

for the first lag state, and

$$\begin{aligned} \lambda\dot{\xi}_{L_l}^{(k)} + \beta_k\xi_{L_l}^{(k)} &= \lambda\dot{\xi}_{L_{(l-1)}}^{(k)} \quad \text{if} \quad p(l, k) - p(l-1, k) = 1 \\ \lambda\dot{\xi}_{L_l}^{(k)} + \beta\xi_{L_l}^{(k)} &= \xi_{L_{(l-1)}}^{(k)} \quad \text{if} \quad p(l, k) - p(l-1, k) = 0 \end{aligned} \tag{27}$$

for the successive lag states from the second lag state onwards, where $2 \le l \le N_L$.

The state-space form of the equations of motion can now be written as $\dot{X} = [A]X$, where $[A]$ is the system dynamic matrix, and has dimension $(2 + N_L)N_\xi$. An explicit expression for the matrix $[A]$ is presented in the Appendix. Both the column-dependent MPPL RFA with a variable number of lag terms per column, as well as the column-independent MPPL RFA, admit similar development for the state-space form of the aircraft equations of motion.

It is clear from Eq. (19) that, for a fixed $N_L(k)$, all the RFAs given by Eq. (19) are equivalent insofar as they lead to the same state vector dimension.

It is interesting to note that a specific multiple pole pure lag RFA representation from the class of MPPL RFAs described by Eq. (19) can be obtained from the SPPL RFA, by considering the case when two pure lag poles have numerical values very close to each other. The approach used in this case is, in essence, similar to that used in Ref. 10. Thus, in the event that two pure lag poles for the same column have essentially the same numerical values, the corresponding matrix coefficients in the RFA representation have numerical values that are very large, of opposite sign, and are differing in magnitude by small amounts. In this case, Eq. (18) takes the form

$$\begin{aligned} \hat{Q}_{jk}(\bar{s}) &= A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 \\ &+ \frac{\bar{s}^3 A_{3_{jk}}}{(\bar{s}+\beta_k-\varepsilon)} - \frac{\bar{s}^3(A_{3_{jk}}-\delta_{jk})}{(\bar{s}+\beta_k+\varepsilon)} \end{aligned} \tag{28}$$

where $|\varepsilon|$ is small in comparison to $|\beta_k|$, and $|\delta_{jk}|$ is small in comparison to $|A_{3_{jk}}|$. Upon rearrangement, the last two terms on the right side of Eq. (28) can be written as

$$2A_{3_{jk}}\varepsilon \frac{\bar{s}^3}{(\bar{s}+\beta_k)^2-\varepsilon^2} + \delta_{jk}\frac{\bar{s}^3}{(\bar{s}+\beta_k+\varepsilon)} \tag{29}$$

Since $|\varepsilon|$ is small in comparison to $|\beta_k|$, Eq. (29) suggests a MPPL RFA representation of the form

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L(k)} A_{(l+2)_{jk}} \frac{\bar{s}^3}{(\bar{s}+\beta_k)^l} \tag{30}$$

It is easily seen that the RFA representation described by Eq. (30) is contained in the class of MPPL RFAs described by Eq. (19), and corresponds to the case when $p(l) = 0$, $l = 1, \ldots, N_L(k)$.

Results generated for the various column-dependent MPPL RFAs described by Eq. (19) showed no differences between the least-squares fit errors as defined by Eq. (10), within the double precision accuracy of the PC-AT 486 computer used, when the lag poles were computed to the fourth decimal place accuracy. Furthermore, the numerical values of the lag poles so computed for all the MPPL RFAs in Eq. (19) were found to be the same. As a consequence, it is conjectured that all the RFAs described by Eq. (19) are equivalent, in that they result in the same numerical values for the lag poles and the same fit error, for the same number of terms under the summation sign. Also, in keeping with the above conjecture, in the following paragraphs, instead of talking about the results obtained for a specific RFA representation from the class of RFAs described by Eqs. (19) or (22), reference is made to the results of the MPPL RFA.

The multiple order pole RFA as given by Eversman and Tewari[10] is

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L} A_{(l+2)_{jk}} \frac{1}{(\bar{s} + \beta)^l} \tag{31}$$

It may be noted that Ref. 10 dealt only with the column-independent formulation. By analogy with the present column-independent formulation, it follows that the RFA in the above equation is also only one of an entire class of multiple pole RFAs, which may be expressed in a generic form as

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{l=1}^{N_L} A_{(l+2)_{jk}} \frac{\bar{s}^{p(l)}}{(\bar{s} + \beta)^l} \tag{32}$$

where $p(l)$ is defined by the column-independent versions of Eqs. (20) and (21).

Equation (31) corresponds to that specific case of Eq. (32), when $p(l) = 0$ for all l. For a fixed N_L, all the RFAs in Eq. (32) lead to the same state vector dimension. It is obvious from Eq. (32) that such a class of multiple pole RFAs is not of a pure lag form. It may further be noted that even the first coefficient matrix in this equation, i.e., $[A_0]$, loses its connotation as the quasisteady approximation to the aerodynamic stiffness, i.e., the static derivative, in comparison to the CLS RFA in Eq. (2).

Results and Discussion

Figures 2 and 3 show the results of the least-squares curve fits obtained using the MPPL RFA, for selected elements from the columns of the aerodynamic matrix that had the largest errors for a given multiplicity. Figures 4 and 5 show the curve fit for the same elements in the column-independent case. It can be seen from these figures that the approximations described by Eqs. (19) and (22) give a fairly good curve fit over the full range of reduced frequency, even with the lowest multiplicity of the pure lag pole, viz., 2. It can also be seen that the accuracy of the fit for a given element increases with an increase in the multiplicity of the associated pure lag pole. It is also seen from these figures that even in the case when no lag poles are used [i.e., only the first three terms in Eqs. (19) and (22) are considered], the curve fit is good in the vicinity of $\bar{k} = 0$, as is expected for a pure lag RFA. With four lag states per column, a very good fit was obtained for all elements of the aerodynamic matrix.

Figure 6 shows a plot of the least-squares fit error as given by Eq. (11) with respect to the multiplicity of the pure lag pole for the column-independent case, and Fig. 7 shows a similar plot of the least-squares fit error as given by Eq. (10) for selected columns in the column-dependent case. In both of these figures, it can be seen that the least-squares fit error decreases monotonically with an increase in the multiplicity of the asssociated pure lag pole. It was also seen that, in general, the fit error decreased approximately exponentially with the multiplicity of the pure lag pole.

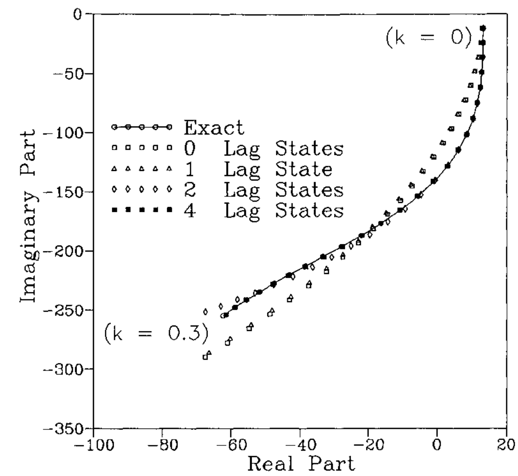

Fig. 2 Column-dependent MPPL RFA curve fits for aerodynamic influence coefficient Q_{22}.

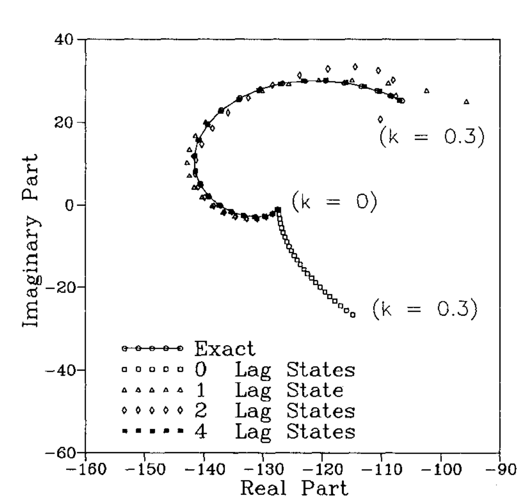

Fig. 3 Column-dependent MPPL RFA curve fits for aerodynamic influence coefficient Q_{27}.

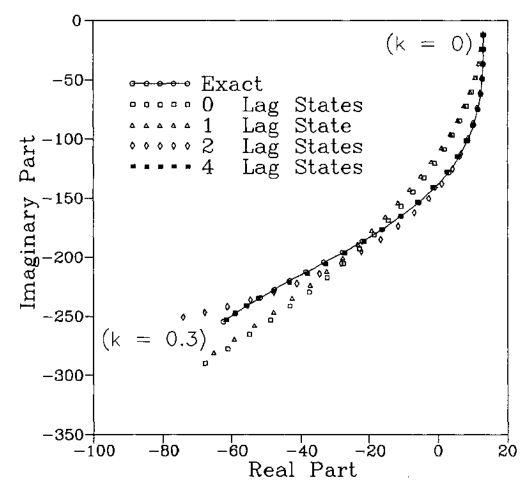

Fig. 4 Column-independent MPPL RFA curve fits for aerodynamic influence coefficient Q_{22}.

As a further validation of the MPPL RFA, the open loop frequency response in terms of the normal acceleration at the sensor locations was computed from the approximate aerodynamic influence coefficients obtained from the MPPL RFA with two lag terms per column. As in the case of the SPPL RFA, the n_z response was computed both from the modal equations and from the state-space equations, and a comparison of the resulting n_z response with the exact values of n_z was carried out. Figure 8 shows the resulting errors in the magnitude and phase of the approximate n_z response with respect to the exact n_z response. It can be seen that both methods of computation of n_z from the approximate aerodynamic influence coefficients lead to exactly the same errors in the n_z response, and that the magnitude of these errors is relatively small. This clearly indicates the superiority of the MPPL RFA over the SPPL RFA, at least for those cases when the optimization results in values of the pure lag poles very close to each other.

Table 2 shows a comparison between the values of the pure lag poles and corresponding fit errors obtained using different

Table 2 Lag values obtained using different pure lag RFAs

Column	Number of lag poles	Multiplicity of each lag pole	Numerical lag values	Fit error value
1	4	1,1,1,1	0.3351 0.3409 0.3479 0.3650	8.4279×10^{-4}
1	2	2,2	0.3390 0.3540	8.4210×10^{-4}
1	1	4	0.3469	8.4139×10^{-4}
5	4	1,1,1,1	0.2230 0.2270 0.2330 0.2450	1.6378×10^{-4}
5	2	3,1	0.2299 0.2390	1.6347×10^{-4}
5	1	4	0.2320	1.6337×10^{-4}
7	4	1,1,1,1	2.4591 2.7791 3.1989 3.8910	1.1067×10^{-1}
7	3	1,1,2	2.5300 2.8100 3.6000	1.1066×10^{-1}
7	2	2,2	2.6801 3.4281	1.1065×10^{-1}
7	2	1,3	2.5900 3.1789	1.1064×10^{-1}
7	1	4	3.0290	1.1063×10^{-1}

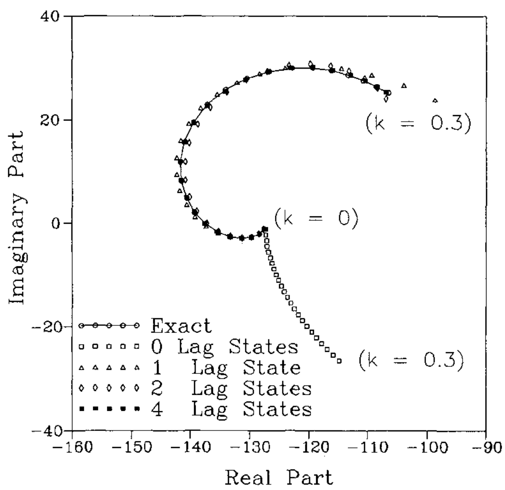

Fig. 5 Column-dependent MPPL RFA curve fits for aerodynamic influence coefficient Q_{27}.

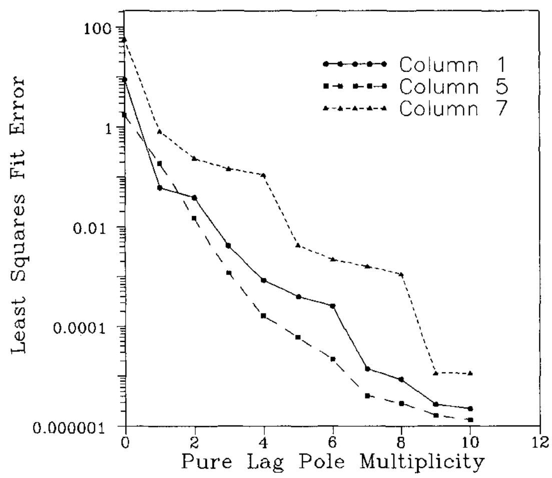

Fig. 7 Variation of least-squares fit error with pure lag pole multiplicity for the column-dependent MPPL RFA.

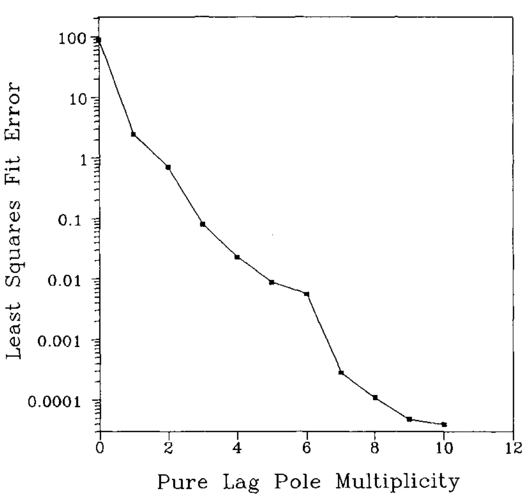

Fig. 6 Variation of least-squares fit error with pure lag pole multiplicity for the column independent MPPL RFA.

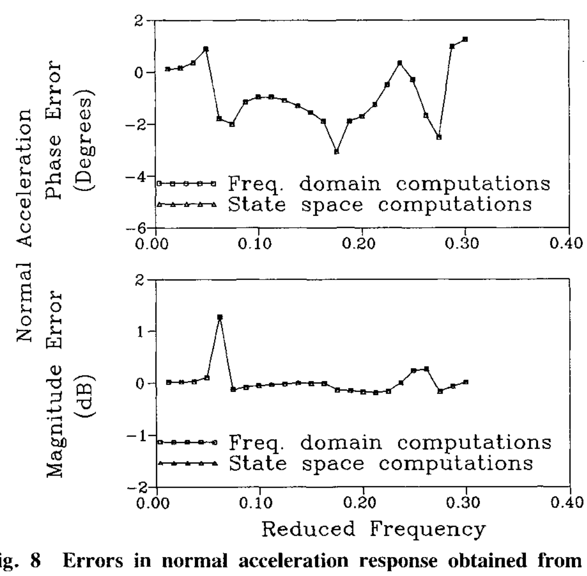

Fig. 8 Errors in normal acceleration response obtained from the column-dependent MPPL RFA using various techniques.

pure lag column-dependent RFAs. The RFAs compared are the MPPL RFA; the SPPL RFA, and a hybrid pure lag (HPL) RFA, which allows for different pure lag poles for a given element, and different multiplicities for each pure lag pole. A generic form for the HPL column-dependent RFA is given as

$$\hat{Q}_{jk}(\bar{s}) = A_{0_{jk}} + A_{1_{jk}}\bar{s} + A_{2_{jk}}\bar{s}^2 + \sum_{p=1}^{r}\sum_{l=1}^{N(k,p)} A_{(m+2)_{jk}} \frac{\bar{s}^3}{(\bar{s} + \beta_{pk})^l} \tag{33}$$

where $m = l + \Sigma_{n=1}^{p-1} N(k, n)$.

Such a form would provide a generalization in the case when the SPPL RFA leads to a set of pure lag poles, with a cluster of poles around each of two or more well-separated values.

A basis for comparison of the three column-dependent RFAs is that their respective contributions to the state vector dimension should be equal. Such a comparison was carried out for those columns of the unsteady aerodynamic matrix, which led to the phenomenon of repeated poles with the use of the SPPL RFA. It was found that for the same contribution to the state vector dimension, all three RFAs led to effectively the same numerical values for the pure lag poles and practically the same fit error values. For example, as shown in Table 2, in the case of column 5, which corresponds to the predominantly wing torsional mode, the SPPL RFA with four simple poles leads to four pure lag values in the range 0.233–0.245. The HPL RFA with two poles, one of multiplicity 3 and the other of multiplicity 1 is seen to lead to two pure lag poles with numerical values 0.2299 and 0.2390, respectively, and the MPPL RFA is seen to lead to a pure lag value of 0.2320 for multiplicity 4. A comparison of the corresponding fit error values shows that the fit errors are almost the same in all three cases, with the MPPL RFA giving the lowest error.

Table 2 shows that the use of the MPPL RFA is also effective in those cases, wherein close values of the pure lag poles were not encountered in the SPPL RFA. This is evident from a comparison of the fit errors of the three approximations for column 7, which corresponds to the control DOF. The SPPL RFA is seen to lead to four well-separated pure lag poles, viz., 2.4591, 2.7791, 3.1989, and 3.8910, whereas the MPPL RFA with multiplicity 4 is seen to lead to a value of 3.0290 for the pure lag pole. Furthermore, Table 2 shows lag pole values for column 7 obtained using the HPL RFA, for varying numbers of lag poles and their multiplicities. These have been chosen such that the resulting contribution to the state vector dimension remains constant and equal to that obtained using the SPPL RFA with four simple poles. It can be seen that the numerical values of the pure lag poles obtained using the HPL and MPPL RFAs lie between the extreme values of the lag poles obtained using the SPPL RFA with four simple poles. A comparison of the fit errors for the three RFAs shows that the fit error in the case of the MPPL and HPL RFAs is slightly lower than that for the SPPL RFA. Thus, the MPPL RFA may be looked upon as a possible generalization of the SPPL RFA.

The advantages of a pure lag RFA representation over a conventional RFA representation have already been outlined earlier. Besides these, the use of a pure lag RFA representation leads to additional computational advantages during the optimization process, in that no additional constraints need to be imposed in order to ensure the accuracy of the fit near $\bar{k} = 0$. In the column-independent MPPL RFA, only one design variable is involved in the optimization process for the complete matrix, irrespective of the number of aerodynamic lag terms used for the curve fit. The corresponding SPPL RFA, on the other hand, involves as many design variables in the optimization process as the number of lag terms used. Thus, the use of the column-independent MPPL RFA leads to a drastic reduction in computation costs, in comparison to the column-independent SPPL RFA. This has also been pointed out by Eversman and Tewari,[10] with a detailed analysis of the computational costs involved. The reduction in computation costs is even more striking in the column-dependent case, since the optimization process needs to be carried out separately for each column of the unsteady aerodynamic matrix.

Conclusions

The computation of lag poles from an existing SPPL RFA by the use of an optimization process is seen to lead to the phenomenon of repeated poles, resulting in an ill-conditioned system dynamic matrix in the state-space form of the aircraft equations of motion. This problem can be overcome by the use of a new class of MPPL RFAs proposed in this article. The new class of MPPL RFAs constitute a consistent generalization of the SPPL RFA, and are characterized by the fact that they preserve both the pure lag form of the approximation, and the state vector dimension for a given value of fit error. The benefits of the new RFA arise both from the pure lag and multiple pole features of the approximation. The pure lag form of the approximation allows for the imposition of fewer constraints on the optimization procedure, in comparison to that adopted in earlier studies, in addition to providing a specific meaning to each coefficient in the RFA. The multiple pole form of the approximation results in a substantial reduction in computation costs for optimization, in addition to giving a well-conditioned system dynamic matrix in the state-space form of the aircraft equations of motion.

Appendix: System Dynamic Matrices

In the case of the column-dependent SPPL RFA with equal number of lag terms per column of the unsteady aerodynamic matrix, the system dynamic matrix $[A]$ can be obtained as $[A] = [E]^{-1}[F]$, where the matrices $[E]$ and $[F]$ are given as

$$[E] = \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ 0 & M - \lambda^2 q_d A_2 & -\lambda q_d A_3 & -\lambda q_d A_4 & \cdots & -\lambda q_d A_{N_L} \\ 0 & \lambda^2 I & -\lambda B_1 & 0 & \cdots & 0 \\ 0 & \lambda^2 I & 0 & -\lambda B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda^2 I & 0 & 0 & \cdots & -\lambda B_{N_L} \end{bmatrix}$$

$$[F] = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ q_d A_0 - K & \lambda q_d A_1 - C & 0 & 0 & \cdots & 0 \\ 0 & 0 & C_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_{N_L} \end{bmatrix}$$

where I is the identity matrix of order $N_\xi \times N_\xi$, $B_l = \text{diag}(\beta_{lk}^2)$ and $C_l = \text{diag}(\beta_{lk}^3)$.

In the case of the column-dependent MPPL RFA with equal number of pure lag terms per column, the system matrix can be written in the form $[A] = [E]^{-1}[F]$, where the matrices

$[E]$ and $[F]$ are in this case given as

$$[E] = \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ 0 & M - q_d\lambda^2 A_2 & -q_d\lambda A_3 & -q_d\lambda A_4 & \cdots & -q_d\lambda A_{N_L} \\ 0 & \lambda^2 I & -\lambda I & 0 & \cdots & 0 \\ 0 & 0 & -\lambda I\delta(2) & \lambda I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda I \end{bmatrix}$$

$$[F] = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ q_d A_0 - K & q_d\lambda A_1 - C & 0 & 0 & \cdots & 0 \\ 0 & 0 & B & 0 & \cdots & 0 \\ 0 & 0 & [1 - \delta(2)]I & -B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -B \end{bmatrix}$$

where $\delta(l) = p(l) - p(l-1)$, $(2 \le l \le N_L)$, and $B = \mathrm{diag}(\beta_k)$.

Acknowledgments

The authors wish to acknowledge the support of the Aeronautical Development Agency (ADA), Bangalore, India, and in particular, that of K. Panda, Scientist, ADA.

References

[1]Roger, K. L., "Airplane Math Modelling Methods for Active Control Design," AGARD CP 228, Aug. 1977.

[2]Abel, I., "An Analytical Design Technique for Predicting the Characteristics of a Flexible Wing Equipped with an Active Flutter Suppression System and Comparison with Wind-Tunnel Data," NASA TP 1367, Feb. 1979.

[3]Vepa, R., "Finite State Modelling of Aeroelastic System," NASA CR 2779, Feb. 1977.

[4]Karpel, M., "Design for Active and Passive Flutter Suppression and Gust Alleviation," NASA CR 3842, Nov. 1981.

[5]Tiffany, S. H., and Adams, W. M., "Nonlinear Programming Extensions to Rational Function Approximation Methods for Unsteady Aerodynamic Forces," NASA TP 2776, July 1988.

[6]Hoadley, S. T., and Karpel, M., "Application of Aeroservoelastic Modelling Using Minimum-State Unsteady Aerodynamic Approximations," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 6, 1991, pp. 1267–1276.

[7]Karpel, M., "Time Domain Aeroservoelastic Modelling Using Weighted Unsteady Aerodynamic Forces," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 1, 1990, pp. 30–37.

[8]Nissim, E., "Reduction of Aerodynamic Augmented States in Active Flutter Suppression Systems," *Journal of Aircraft*, Vol. 28, No. 1, 1991, pp. 82–93.

[9]Karpel, M., "Size Reduction Techniques for the Determination of Efficient Aeroservoelastic Models," *Control and Dynamic Systems*, edited by C. T. Leondes, Vol. 54, Academic Press, San Diego, CA, 1992, pp. 263–295.

[10]Eversman, W. A., and Tewari, A., "Consistent Rational Function Approximation for Unsteady Aerodynamics," *Journal of Aircraft*, Vol. 28, No. 5, 1991, pp. 545–552.

[11]Suryanarayan, S., Panda, K., and Prakash, B. G., "Development of a Pure Lag Expression for the Time Domain Representation of Modal Unsteady Airloads," *Journal of Aircraft* (to be published).

[12]Rao, S. S., *Optimization Theory and Applications*, 2nd ed., Wiley Eastern Ltd., New Delhi, India, 1984.